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On the modulation instability of nonlinear Schrödinger equations

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Abstract. According to the modulation instability of the nonlinear and derivative nonlinear Schrodinger equations, a wave profile may decay into solitons The distribution function and the power spectrum of the solitons are calculated from conservation laws

1. Introduction

The nonlinear Schrödinger equation

$$iq_t + q_{xx} + 2|q|^2 q = 0 \tag{1}$$

exhibits a modulation instability (Benjamin and Feir 1967), i.e. small perturbations of a wave exponentially grow in the course of time. A numerical solution of the nonlinear Schrödinger equation (Shen and Nicholson 1987) shows that a small stochastic perturbation of a constant profile leads to the formation of solitons. The calculated spectrum qualitatively agrees with a simple analytic approximation assuming equal-amplitude, randomly spaced, zero-speed non-overlapping solitons. This approximation, however, fulfills only the first of an infinite number of conservation laws. Recently, a distribution function of the amplitudes and velocities of solitons has been calculated which satisfies all polynomial conservation laws (Dawson and Fontán 1989). After a short rederivation of this distribution function in section 2, we calculate the power spectrum of randomly spaced non-overlapping solitons from the distribution function and find good agreement with the numerical calculations (Shen and Nicholson 1987), except for a missing smooth part with small wavenumbers.

Guided by this success, in section 3 we extend these calculations to the derivative nonlinear Schrödinger equation

$$iq_t + q_{xx} \pm i(|q|^2 q)_x = 0 \tag{2}$$

which also exhibits a modulation instability (Mio *et al* 1976, Mjølhus and Wyller 1986, Mann 1988) under certain conditions, whereby solitons arise from a weakly disturbed wave profile. A short recalculation of the distribution function shows that Dawson and Fontán's (1989) result is valid only for normal solitons, whereas an additional term arises for anomalous solitons. The distribution function is used to calculate the power spectrum of the solitons. The power spectrum vanishes for zero wavenumber and is somewhat broader than a typical one-soliton spectrum.

Finally, the results are summarized and discussed in section 4.

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2. Nonlinear Schrödinger equation

The nonlinear Schrödinger equation is related to the spectral problem (Zakharov and Shabat 1971)

$$\varphi_x = -\mathrm{i}\lambda\varphi + q\chi \qquad \chi_x = -q^*\varphi + \mathrm{i}\lambda\chi. \tag{3}$$

$\binom{\varphi}{\chi}$ is the two-component spectral function, and λ the spectral parameter. Eliminating χ yields

$$\varphi_{xx} - (\varphi_x + \mathrm{i}\lambda\varphi)\frac{q_x}{q} = -\lambda^2\varphi - |q|^2\varphi. \tag{4}$$

With the ansatz

$$\varphi = \mathrm{e}^{-\mathrm{i}\lambda x + \phi(x)} \tag{5}$$

we obtain

$$2\mathrm{i}\lambda\phi_x = |q|^2 + q\left(\frac{\phi_x}{q}\right)_x + \phi_x^2 \tag{6}$$

If $\phi(x=-\infty)=0$ then $\phi(x=\infty)=\int_{-\infty}^{\infty}\phi_x(x)\,\mathrm{d}x$ is independent of time for all λ. When this function is written as a series in powers of $(1/2\mathrm{i}\lambda)$, the coefficients are the polynomial conserved quantities of the nonlinear Schrodinger equation (Zakharov and Shabat 1971).

Let us calculate $\phi(\infty)$ for the soliton solution

$$q = \frac{u}{\cosh u\xi}\exp\left\{\mathrm{i}\left[\frac{v}{2}\xi + \left(\frac{v^2}{4} + u^2\right)t\right]\right\} \qquad \xi = x - vt \tag{7}$$

of (1) and the wave profile

$$q = q_0\,\mathrm{e}^{\mathrm{i}(kx-\omega t)} \qquad \omega = k^2 - 2q_0^2 \tag{8}$$

which is unstable for a small perturbation with a wavenumber $\kappa < 2q_0$, when $q_\kappa \sim \exp(\kappa\sqrt{4q_0^2-\kappa^2}\,t)$. Solving (6) for the soliton (7) we obtain

$$\phi(\xi) = \ln\frac{\lambda + v/4 - (\mathrm{i}u/2)\tanh u\xi}{\lambda + v/4 + \mathrm{i}u/2} \tag{9}$$

$$\phi(\infty) = \ln\frac{\lambda + v/4 - \mathrm{i}u/2}{\lambda + v/4 + \mathrm{i}u/2} = -2\mathrm{i}\tan^{-1}\frac{u}{2(\lambda + v/4)}. \tag{10}$$

For the wave profile (8) we have from (6)

$$\phi_x = \mathrm{i}\left[\left(\lambda + \frac{k}{2}\right) - \operatorname{sign}\left(\lambda + \frac{k}{2}\right)\sqrt{\left(\lambda + \frac{k}{2}\right)^2 + q_0^2}\right]. \tag{11}$$

The plane wave is assumed to evolve into an ensemble of solitons with the distribution function $\rho(u, v)$, where $\rho(u, v)\,\mathrm{d}u\,\mathrm{d}v$ is the number of solitons with amplitude u in an interval $\mathrm{d}u$, and velocity v in an interval $\mathrm{d}v$. The conserved function for the solitons is additive and must be equal to the conserved function for the initial plane wave. This condition leads to an integral equation for determining the distribution function $\rho(u, v)$ of the solitons, i.e.

$$\int \phi(\infty; \lambda; u, v)\rho(u, v; q_0, k)\,\mathrm{d}u\,\mathrm{d}v = \phi_x(\lambda; q_0, k)L \tag{12}$$

where $\phi(\infty, \lambda; u, v)$ and $\phi_x(\lambda; q_0, k)$ are given by (10) and (11), and L is the length of some large interval containing the initial plane wave. Equation (12) is solved by

$$\rho(u, v) = \frac{L}{2\pi} \frac{u}{\sqrt{4q_0^2 - u^2}} \delta(v - 2k) \tag{13}$$

(Dawson and Fontán 1989) All solitons emerging from the wave profile (8) have amplitudes $u < 2q_0$ and a unique velocity $v = 2k$. The total number of solitons becomes

$$\int \rho(u, v)\, \mathrm{d}u\, \mathrm{d}v = \frac{q_0 L}{\pi} \tag{14}$$

and the total width of all solitons per unit length is

$$\frac{1}{L} \int \frac{2}{u} \rho(u, v)\, \mathrm{d}u\, \mathrm{d}v = \tfrac{1}{2} \tag{15}$$

Thus only half the interval is crowded with solitons which only weakly overlap.

The power spectrum of the solitons derives from the Fourier transform of a soliton (7):

$$\hat{q}(K) = \frac{1}{L} \int_{-\infty}^{\infty} q(\xi, t=0)\, \mathrm{e}^{-\mathrm{i}K\xi}\, \mathrm{d}\xi \tag{16}$$

$$= \frac{\pi}{L \cosh[\pi(K - v/2)/2u]}. \tag{17}$$

For randomly spaced, non-overlapping solitons with the distribution function (13) the power spectrum becomes

$$P(K) = \int \hat{q}^2(K; u, v) \rho(u, v)\, \mathrm{d}u\, \mathrm{d}v \tag{18}$$

$$= \frac{\pi q_0}{L} \int_0^{\pi/2} \frac{\sin\varphi}{\cosh^2[\pi(K - k)/4q_0 \sin\varphi]}\, \mathrm{d}\varphi. \tag{19}$$

Figure 1 shows the power spectrum (19) together with the results $P(K) = N\hat{q}^2(K)$ from two simple one-soliton group approximations. In the approximation of Shen and Nicholson (1987), the number of solitons is assumed to be $N = \kappa L/2$, where $\kappa = \sqrt{2}\, q_0$ is the wavenumber with maximum growth rate, and the common amplitude $u = \pi q_0/\sqrt{2}$ and velocity $v = 2k$ of all solitons are determined from the first two polynomial conservation laws. A better one-soliton group approximation is obtained from the first three conservation laws which yield $N = q_0 L/2\sqrt{3}$ solitons with amplitude $u = \sqrt{3}\, q_0$ and velocity $v = 2k$. This approximation scarcely differs from (19) for large wavenumbers, but yields a smaller power for small wave numbers. The power spectrum (19) also agrees well with the numerical solution of the nonlinear Schrödinger equation (Shen and Nicholson 1987) for a constant profile ($k = 0$) and large K, but still comes out too small for small K. Thus we conclude that a constant profile approximately decays into different solitons plus some smooth background with small wavenumbers K.

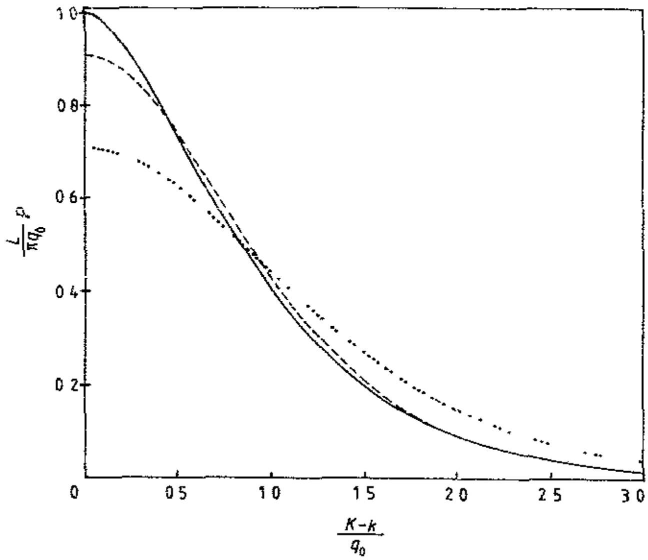

Figure 1. Power spectrum of solitons of the nonlinear Schrodinger equation as calculated from the Shen–Nicholson approximation ( ), an improved one-soliton group approximation (- - - -), and from the soliton distribution function (——)

3. Derivative nonlinear Schrödinger equation

The derivative nonlinear Schrödinger equation (2) is related to the spectral problem (Kaup and Newell 1978)

$$\varphi_x = -i\lambda\varphi + q\sqrt{\lambda}\,\chi \qquad \chi_x = \pm q^*\sqrt{\lambda}\,\varphi + i\lambda\chi \tag{20}$$

which is similar to the Zakharov–Shabat spectral problem (3) for the nonlinear Schrödinger equation (1). Thus the distribution function of solitons emerging from a wave profile and the power spectrum of solitons can be calculated along the same lines as described in the previous section. Elimination of χ in (20) yields

$$\varphi_{xx} - (\varphi_x + i\lambda\varphi)\frac{q_x}{q} = -\lambda^2\varphi \pm \lambda|q|^2\varphi \tag{21}$$

With the ansatz (5) we obtain

$$(2i\phi_x \pm |q|^2)\lambda = q\left(\frac{\phi_x}{q}\right)_x + \phi_x^2. \tag{22}$$

Solving (22) for the soliton solution of (2) (Mjølhus and Wyller 1986, Mann 1988)

$$q = \frac{u}{\sqrt{(1-W)\cosh 4b\xi + W}} \times \exp\left[i\left(\pm 3\tan^{-1}(\sqrt{1-2W}\tanh 2b\xi) + \frac{v}{2}\xi + \frac{1}{16}(u^2 \pm 2v)^2 t\right)\right] \tag{23}$$

$$\xi = x - vt \qquad b = \frac{u}{8}\sqrt{u^2 \pm 4v} \qquad W = \frac{\pm 2v}{u^2 \pm 4v} \tag{24}$$

yields

$$\phi(\xi)=\ln\frac{(v/4)^2+b^2+\lambda\{v/4+\mathrm{i}[b(1-W)\sinh 4b\xi\mp \mathrm{i}u^2/8]/[(1-W)\cosh 4b\xi+W]\}}{(v/4)^2+b^2+\lambda(v/4-\mathrm{i}b)}. \tag{25}$$

The conserved function of λ becomes

$$\phi(\infty)=\ln\left(\frac{\lambda+v/4-\mathrm{i}b}{\lambda+v/4+\mathrm{i}b}\,\frac{v/4+\mathrm{i}b}{v/4-\mathrm{i}b}\right)$$
$$=2\mathrm{i}\left(-\tan^{-1}\frac{b}{\lambda+v/4}+\tan^{-1}\frac{4b}{v}\pm\varepsilon(\mp v)\pi\right) \tag{26}$$

where $\varepsilon(v)=\frac{1}{2}[1+\mathrm{sign}(v)]$ is the unit step function. For the wave profile

$$q=q_0\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}\qquad \omega=k^2\mp kq_0^2 \tag{27}$$

we obtain

$$\phi_x=\mathrm{i}\left[\left(\lambda+\frac{k}{2}\right)-\mathrm{sign}\left(\lambda+\frac{k}{2}\right)\sqrt{\left(\lambda+\frac{k}{2}\right)^2\mp q_0^2\lambda}\right] \tag{28}$$

The wave profile decays into solitons only if $\pm k>q_0^2/2$, when small perturbations with wavenumbers $\kappa<\sqrt{\pm 2k-q_0^2}\,q_0$ grow as $q_\kappa\sim\exp[\kappa\sqrt{(\pm 2k-q_0^2)q_0^2-\kappa^2}\,t]$. In this case the integral equation (12) for the soliton distribution function is solved by

$$\rho(u,v)=\frac{L}{\pi}\left(\frac{b}{\sqrt{Q^2-b^2}}\delta(v-v_0)+\varepsilon(\mp v_0)\delta(b)\rho_1(v)\right)\frac{\mathrm{d}b}{\mathrm{d}u} \tag{29}$$

where

$$Q=q_0\sqrt{\frac{1}{2}\left(\pm k-\frac{q_0^2}{2}\right)}\qquad v_0=\pm 2(\pm k-q_0^2) \tag{30}$$

$$\int\varepsilon(\mp v)\rho_1(v)\,\mathrm{d}v=\pm\frac{k}{2}-Q>0 \tag{31}$$

If $\pm v_0>0$ the last term in (29) vanishes, and normal solitons with limited amplitudes ($b<Q$) and a unique velocity v_0 occur. In this case the distribution function has been derived by Dawson and Fontán (1989) by transforming the Kaup–Newell spectral problem (20) to the Zakharov–Shabat problem (3). However, this transformation does not adequately take into account the first polynomial conserved quantity

$$\mp 2\mathrm{i}\phi(\xi=\infty;\lambda\to\infty)=\int_{-\infty}^{\infty}|q|^2\,\mathrm{d}\xi \tag{32}$$

This quantity directly derives from (22) in the limit $\lambda\to\infty$, whereas in the Zakharov–Shabat case (6) we have $\phi(\xi=\infty;\lambda\to\infty)=0$. Since (32) is a positive continuous function of v, a step-function term occurs in (26) which gives rise to the second term in the distribution function (29). Thus for $\pm v_0<0$ not only anomalous solitons with the velocity v_0 are formed, but also some solitons with $u^2=\mp 4v$ ($b=0$), for which according to (26) all polynomial conserved quantities vanish except the first one. These solitons are called algebraic since (23) leads to

$$q=2\sqrt{\frac{\mp v}{1+v^2\xi^2}}\exp\left[\mathrm{i}\left(-3\tan^{-1}(v\xi)+\frac{v}{2}\xi+\frac{v^2}{4}t\right)\right] \tag{33}$$

in the limit $b \to 0$ Algebraic solitons, however, have infinite width and may decay into radiation under small perturbations (Kaup and Newell 1978). Therefore a wave profile with $\pm v_0 < 0$ cannot evolve into an ensemble of stable solitons with only small overlap.

Let us confine ourselves to the case $\pm v_0 > 0$, when the soliton distribution (29) is

$$\rho(u, v) = \frac{L}{\pi} \frac{b}{\sqrt{Q^2 - b^2}} \delta(v - v_0) \frac{\mathrm{d}b}{\mathrm{d}u}. \tag{34}$$

The total number of solitons is

$$\int \rho(u, v)\, \mathrm{d}u\, \mathrm{d}v = \frac{QL}{\pi} \tag{35}$$

and the total width of all solitons per unit length becomes

$$\frac{1}{L} \int \frac{1}{b} \rho(u, v)\, \mathrm{d}u\, \mathrm{d}v = \tfrac{1}{2} \tag{36}$$

as in (15)

The Fourier transform (16) of a soliton (23) is

$$\hat{q}(K) = \frac{u}{bL} f\left(W, \pm \frac{K - v/2}{2b} \right) \tag{37}$$

with

$$f(W, K_1) = \int_0^\infty \frac{\cos[3 \tan^{-1}(\sqrt{1 - 2W} \tanh \xi) - K_1 \xi]}{\sqrt{(1 - W) \cosh 2\xi + W}}\, \mathrm{d}\xi. \tag{38}$$

The power spectrum (18) is calculated from the Fourier transform (37) and the distribution function (34) of the solitons. Using (24) to write

$$u = \sqrt{8b \frac{W(v_1)}{v_1}} \qquad W(v_1) = \frac{v_1}{v_1 + \sqrt{1 + v_1^2}} \tag{39}$$

in terms of $v_1 = \pm v/4b$ we obtain the power spectrum

$$P(K) = \frac{8}{\pi L} \int_0^{\pi/2} \frac{W(v_1(\varphi))}{v_1(\varphi)} f^2\left(W(v_1(\varphi)), \pm \frac{K - v_0/2}{2Q \sin \varphi} \right) \mathrm{d}\varphi \tag{40}$$

where $v_1(\varphi) = \pm v_0/4Q \sin \varphi$.

This result may be compared with a simple one-soliton group approximation $P(K) = N\hat{q}^2(K)$, for which the first three conservation laws yield $N = Q^2L/4b$ solitons with a common amplitude u (39) and velocity $v = v_0$ (30), where b must be determined from

$$\frac{\pi}{2} \mp \tan^{-1} \frac{v}{4b} = \frac{q_0^2}{Q^2} b. \tag{41}$$

All soliton spectra vanish for $|K| \to \infty$ and $K \to 0$ since

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} q(\xi)\, \mathrm{d}\xi = \frac{\mathrm{i}}{16} (u^2 \pm 2v)^2 \int_{-\infty}^{\infty} q(\xi)\, \mathrm{d}\xi = 0 \tag{42}$$

according to (2) and (23), and have maxima for finite $|K|$.

Let us consider the special case $\pm k = q_0^2$ when $Q = q_0^2/2$ and $v_0 = 0$ according to (30). The one-soliton group approximation yields $b = \pi q_0^2/8$ from (41), i.e. $N = q_0^2 L/2$

solitons with amplitude $u=\sqrt{\pi}\,q_0$ and velocity $v=0$. Then the power spectrum (40) becomes

$$P(K)=\frac{8}{\pi L}\int_0^{\pi/2} f^2\left(0,\pm\frac{K}{q_0^2\sin\varphi}\right)\mathrm{d}\varphi \tag{43}$$

for the soliton distribution, and

$$P(K)=\frac{32}{\pi^2 L} f^2\left(0,\pm\frac{4K}{\pi q_0^2}\right) \tag{44}$$

in the one-soliton approximation As shown in figure 2, both spectra are rather similar. The integration over the soliton distribution in (43), however, leads to a somewhat shifted and broadened maximum as compared with the one-soliton approximation (44).

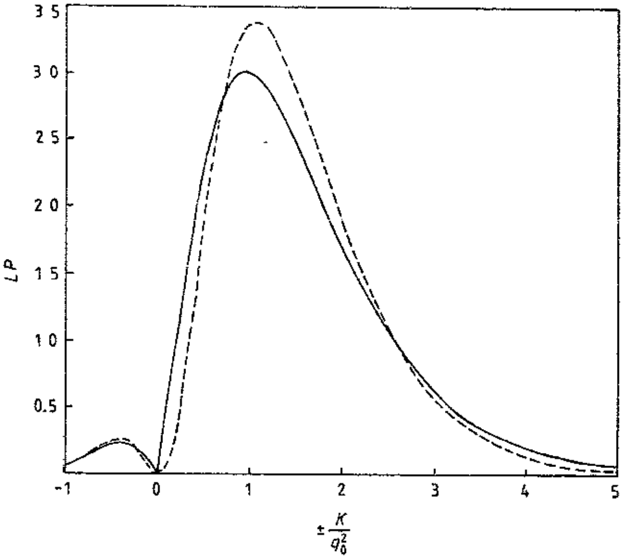

Figure 2. Power spectrum of zero-speed solitons ($\pm k=q_0^2$) of the derivative nonlinear Schrödinger equation as calculated from a one-soliton group approximation (----), and from the soliton distribution function (——)

4. Summary

We have studied the modulation instability of a wave profile for the nonlinear and derivative nonlinear Schrödinger equations. Strictly speaking, the evolution of a wave profile depends on the detailed shape of the initial small perturbation. We discuss here, however, a typical evolution of the wave profile into an ensemble of stable solitons under small stochastic perturbations. In this case the distribution function of the solitons have been determined from the polynomial conserved quantities by Dawson and Fontán (1989). We confirm their result except for $q_0^2/2<\pm k<q_0^2$ in the derivative nonlinear Schrödinger case, when the first conservation law can be satisfied only with additional algebraic solitons.

The power spectrum of the solitons can be calculated from their distribution function, provided the solitons are randomly distributed and do not overlap. This condition is only approximately met, since half of the available space is occupied by the solitons For the nonlinear Schrödinger equation the spectrum qualitatively agrees with a numerical solution of this equation (Shen and Nicholson 1987). For the derivative nonlinear Schrödinger equation, the distribution function is compatible with numerical solutions involving only few solitons, but extending the numerical calculations to a large number of solitons would appear to be difficult (Dawson and Fontán 1988, 1989).

The power spectrum of the derivative nonlinear Schrödinger equation exhibits a maximum for finite K Such maximum is observed in the power spectrum of the Alfvénic turbulence upstream of the earth's bow shock, which is related to the derivative nonlinear Schrödinger equation (Möbius *et al* 1987, Mann 1988, 1990).

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